

FUNCTIONS CONCERNED WITH DIVISORS OF ORDER r

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ABSTRACT. N. Minculete has introduced a concept of divisors of order r : integer $d = p_1^{b_1} \cdots p_k^{b_k}$ is called a divisor of order r of $n = p_1^{a_1} \cdots p_k^{a_k}$ if $d \mid n$ and $b_j \in \{r, a_j\}$ for $j = 1, \dots, k$. One can consider respective divisor function $\tau^{(r)}$ and sum-of-divisors function $\sigma^{(r)}$.

In the present paper we investigate the asymptotic behaviour of

$$\sum_{n \leq x} \tau^{(r)}(n) \text{ and } \sum_{n \leq x} \sigma^{(r)}(n)$$

and improve several results of [10] and [11]. We also provide conditional estimates under Riemann hypothesis.

1. INTRODUCTION

Recently N. Minculete in his PhD Thesis [10], devoted to the functions using exponential divisors, and in further paper [11] introduced a concept of *divisors of order r* : integer $d = p_1^{b_1} \cdots p_k^{b_k}$ is called a divisor of order r of number $n = p_1^{a_1} \cdots p_k^{a_k}$ if d divides n in the usual sense and $b_j \in \{r, a_j\}$ for $j = 1, \dots, k$. We also suppose that 1 is a divisor of any order of itself (but not of any other number). Let us denote respective divisor and sum-of-divisor functions as $\tau^{(r)}$ and $\sigma^{(r)}$. These functions are multiplicative and

$$(1) \quad \tau^{(r)}(p^a) = \begin{cases} 1, & a \leq r, \\ 2, & a > r. \end{cases}$$

$$(2) \quad \sigma^{(r)}(p^a) = \begin{cases} p^a, & a \leq r, \\ p^a + p^r, & a > r. \end{cases}$$

In a special case of $r = 0$ we get well-studied *unitary divisors*. For example, it was proved in [3] that

$$(3) \quad \sum_{n \leq x} \tau^{(0)}(n) = \frac{x}{\zeta(2)} \left(\log x + 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)} \right) + O(x^{1/2}).$$

(under Riemann hypothesis error term is $O(x^{221/608+\varepsilon})$ due to [7]) and in [14] it was proved that

$$(4) \quad \sum_{n \leq x} \sigma^{(0)}(n) = \frac{\pi^2 x^2}{12\zeta(3)} + O(x \log^{5/3} x).$$

In another special case of $r = 1$ we get so-called by Minculete *exponential semiproper divisors* and denote $\tau^{(e)s} := \tau^{(1)}$, $\sigma^{(e)s} := \sigma^{(1)}$. An integer d is an exponential semiproper divisor of n if $\ker d = \ker n$ and $(d/\ker n, n/d) = 1$, where $\ker n = \prod_{p \mid n} p$.

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Minculete proved in [10, (3.1.17–19)] that

$$(5) \quad \limsup_{n \rightarrow \infty} \frac{\log \tau^{(r)}(n) \log \log n}{\log n} = \frac{\log 2}{r+1},$$

$$(6) \quad \sum_{n \leq x} \tau^{(r)}(n) = \frac{\zeta(r+1)}{\zeta(2r+2)} x + Ax^{1/(r+1)} + O(x^{1/(r+2)+\varepsilon}),$$

$$(7) \quad \limsup_{n \rightarrow \infty} \frac{\sigma^{(r)}(n)}{n \log \log n} = \frac{6e^\gamma}{\pi^2}.$$

In the present paper we improve the error term in (6) and establish asymptotic formulas for $\sum_{n \leq x} \sigma^{(r)}(n)$ with O - and Ω -estimates of the error term.

2. NOTATION

In asymptotic relations we use \sim , \asymp , Landau symbols O and o , big omegas Ω and Ω_\pm , Vinogradov symbols \ll and \gg in their usual meanings. All asymptotic relations are given as an argument tends to the infinity.

Letter p with or without indexes denote rational prime.

As usual $\zeta(s)$ is Riemann zeta-function. For complex s we denote $\sigma := \Re s$ and $t := \Im s$.

We use abbreviations $\log \log x := \log \log x$, $\text{llog } x := \log \log \log x$.

Letter γ denotes Euler–Mascheroni constant, $\gamma \approx 0.577$.

Everywhere $\varepsilon > 0$ is an arbitrarily small number (not always the same even in one equation).

We write $f \star g$ for Dirichlet convolution: $(f \star g)(n) = \sum_{d|n} f(d)g(n/d)$.

Function $\ker: \mathbb{N} \rightarrow \mathbb{N}$ stands for $\ker n = \prod_{p|n} p$.

For a set A notation $\#A$ means the cardinality of A .

3. PRELIMINARY ESTIMATES

Consider

$$\tau(a, b; n) = \sum_{k^a l^b = n} 1, \quad T(a, b; x) = \sum_{n \leq x} \tau(a, b; n), \quad 1 \leq a \leq b.$$

One can directly check that

$$\sum_{n=1}^{\infty} \frac{\tau(a, b; n)}{n^s} = \zeta(as)\zeta(bs), \quad \sigma > 1$$

Lemma 1.

$$T(a, b; x) = H(a, b; x) + \Delta(a, b; x)$$

where

$$H(a, b; x) = \begin{cases} \zeta(b/a)x^{1/a} + \zeta(a/b)x^{1/b}, & 1 \leq a < b, \\ x^{1/a} \log x + (2\gamma - 1)x^{1/a}, & a = b, \end{cases}$$

and

$$x^{1/2(a+b)} \ll \Delta(a, b; x) \ll \begin{cases} x^{1/(2a+b)} & 1 \leq a < b, \\ x^{1/3a} \log x & a = b. \end{cases}$$

Proof. See [8, Th. 5.1, Th. 5.3, Th. 5.8]. ■

In fact $\Delta(a, b; x)$ can be estimated more precisely. For our goals we are primarily interested in the behaviour of $\Delta(1, b; x)$. Let us suppose that

$$(8) \quad \Delta(1, b; x) \ll x_b^\theta \log^{\theta'_b} x,$$

then due to [8, Th. 5.11] we can choose

$$\theta_b = \frac{1}{b + 7/2}, \quad \theta'_b = 1, \quad b \geq 7.$$

Estimates for $b \leq 16$ are given in Table 1. Estimate for $b = 1$ belongs to Huxley [5], and estimate for $b = 2$ belongs to Graham and Kolesnik [4]. We have found no references on the best known results for $b \geq 3$, so we calculated them with the use of [8, Th. 5.11, Th. 5.12] selecting appropriate exponent pairs carefully. It seems that some of this estimates may be new.

Lemma 2. *Let α and β be positive real numbers with $\beta + 1 \leq \alpha$. Then*

$$\sum_{mn^\alpha \leq x} mn^\beta = \frac{\zeta(2\alpha - \beta)}{2} x^2 + \mathcal{D}(\alpha, \beta; x), \quad \mathcal{D}(\alpha, \beta; x) \ll \begin{cases} x \log^{2/3} x, & \beta + 1 = \alpha, \\ x, & \beta + 1 < \alpha. \end{cases}$$

Proof. See [13, Th. 1]. ■

For $k > 0$ one can define a multiplicative function μ_k implicitly by

$$\sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s} = \frac{1}{\zeta(ks)}, \quad \sigma > 1.$$

b	θ_b	θ'_b	Exponent pair or reference
1	$131/416 + \varepsilon \approx 0.314904$	0	[5]
2	$1057/4785 + \varepsilon \approx 0.220899$	0	[4]
3	$1486/8647 + \varepsilon \approx 0.171852$	0	$AB(AB)^2H$
4	$1448/10331 + \varepsilon \approx 0.140161$	0	AH
5	$71318556275/587475333596 + \varepsilon \approx 0.121398$	0	$(A^2B)^2(AB)^2A^3B^6(AB)^2BA^3BH$
6	$669/6305 \approx 0.106106$	1	$(A^2B)^3(AB)^3A^4BI$
7	$338866613/3586241504 \approx 0.094491$	2	$A^2B^3(AB)^2A^3B^6AB(A^2B)^2ABI$
8	$2000836147/23452726172 \approx 0.085314$	1	$A(AB)^4(A^3B)^6A^3BI$
9	$372854090/4786779707 \approx 0.077892$	2	$A^2B^2(AB)^5(AB)^3BA^3B^6ABI$
10	$150509/2096993 + \varepsilon \approx 0.071774$	0	$(A^2B^2)^3ABH$
11	$1048/15811 + \varepsilon \approx 0.066283$	0	A^2H
12	$64/1037 + \varepsilon \approx 0.061716$	0	A^2H
13	$2516635/43324033 + \varepsilon \approx 0.058089$	0	$A^3BA^3BA^2BA^4B(AB)^2H$
14	$75/1373 \approx 0.054625$	1	$A^2(AB)^2BA^3BI$
15	$13514730527/262064292044 + \varepsilon \approx 0.051570$	0	$A(A^2B)^3A^4B^7A^3BBA^4BH$
16	$15/307 \approx 0.048860$	1	$A^3BA^2BA^4BI$

TABLE 1. Values of θ_b and θ'_b in (8) for $b \leq 16$. Exponent pairs are written in terms of A - and B -processes [8, Th. 2.12, 2.13]. We abbreviate $B := BA$. Here $I = (0, 1)$ and $H = (32/205 + \varepsilon, 269/410 + \varepsilon)$ is Huxley exponent pair from [5].

So $\mu_k(n^k) = \mu(n)$ and $\mu_k(m) = 0$ for all other arguments. Trivially $\mu_1 \equiv \mu$. Then

$$M_k(x) := \sum_{n \leq x} \mu_k(n) = \sum_{n \leq x^{1/k}} \mu(n) \ll x^{1/k} \exp(-CN(x)),$$

where $C > 0$, $N(x) = \log^{3/5} x \log^{-1/5} x$. See [6, Th. 12.7] for the proof of the last estimate. Assuming Riemann hypothesis (RH) we get much better result

$$M_k(x) \ll x^{1/2k+\varepsilon} \quad [15, \text{Th. 14.25 (C)}].$$

Lemma 3. *Let $K \in \mathbb{N}$, $J \in \mathbb{N} \cup \{0\}$, $m_1 \leq \dots \leq m_K$, $n_1 \leq \dots \leq n_J$, where all $m_k, n_j \in \mathbb{N}$, and suppose that*

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \frac{\zeta(m_1 s) \cdots \zeta(m_K s)}{\zeta(n_1 s) \cdots \zeta(n_J s)}.$$

Let

$$\alpha = \frac{K-1}{2 \sum_{k=1}^K m_k}.$$

If $1/\alpha < 2n_j$ for all $j = 1, \dots, J$ then for arbitrary $H(x)$ of the form

$$H(x) = \sum_{i=1}^I x^{\beta_i} P_i(\log x), \quad \beta_i \in \mathbb{C}, \quad \alpha < \Re \beta_i \leq 1, \quad P_i \text{ are polynomials,}$$

we have

$$\sum_{n \leq x} a_n = H(x) + \Omega(x^\alpha).$$

Proof. This is a simplified version of [9, Th. 2]. ■

4. ASYMPTOTIC PROPERTIES OF $\sum \tau^{(r)}(n)$

Lemma 4. *Let $F_r(s)$ be Dirichlet series for $\tau^{(r)}$:*

$$F_r(s) := \sum_{n=1}^{\infty} \frac{\tau^{(r)}(n)}{n^s}.$$

Then

$$(9) \quad F_r(s) = \frac{\zeta(s) \zeta((r+1)s)}{\zeta((2r+2)s)}, \quad \sigma > 1.$$

Proof. Let us transform Bell series for $\tau^{(r)}$:

$$\begin{aligned} \tau_p^{(r)}(x) &= \sum_{k=0}^{\infty} \tau^{(r)}(p^k) x^k = \sum_{k=0}^r x^k + 2 \sum_{k>r} x^k = \sum_{k=0}^{\infty} x^k + \sum_{k>r} x^k = \\ &= (1 + x^{r+1}) \sum_{k=0}^{\infty} x^k = \frac{1 + x^{r+1}}{1 - x} = \frac{1 - x^{2r+2}}{(1 - x)(1 - x^{r+1})}. \end{aligned}$$

The representation of F_r in the form of an infinite product by p completes the proof:

$$F_r(s) = \prod_p \tau_p^{(r)}(p^{-s}) = \prod_p \frac{1 - p^{-(2r+2)s}}{(1 - p^{-s})(1 - p^{-(r+1)s})} = \frac{\zeta(s) \zeta((r+1)s)}{\zeta((2r+2)s)}.$$
■

It follows from (9) that

$$(10) \quad \tau^{(r)} = \tau(1, r+1; \cdot) \star \mu_{2r+2}$$

Theorem 1. *If Δ is estimated as in (8) then for $r > 0$*

$$\sum_{n \leq x} \tau^{(r)}(n) = Ax + Bx^{1/(r+1)} + \mathcal{E}_{r+1}(x), \quad \mathcal{E}_r(x) = O\left(x^{\max(\theta_r, 1/2r)} \log^{\theta'_r} x\right),$$

where constants A and B are specified below in (11).

Proof. Taking into account (10) we have for $r > 0$

$$\begin{aligned} \sum_{n \leq x} \tau^{(r)}(n) &= \sum_{n \leq x} \mu_{2r+2}(n) T(1, r+1; x/n) = \zeta(r+1)x \sum_{n \leq x} \frac{\mu_{2r+2}(n)}{n} + \\ &+ \zeta(1/(r+1))x^{1/(r+1)} \sum_{n \leq x} \frac{\mu_{2r+2}(n)}{n^{1/(r+1)}} + \sum_{n \leq x} \mu_{2r+2}(n) \Delta(1, r+1, x/n). \end{aligned}$$

But for $s \geq 1/k$

$$\sum_{n \leq x} \frac{\mu_k(n)}{n^s} = \frac{1}{\zeta(ks)} - \sum_{n > x} \frac{\mu_k(n)}{n^s} = \frac{1}{\zeta(ks)} + O(x^{1/k-s})$$

and

$$\begin{aligned} \sum_{n \leq x} \mu_{2k}(n) \Delta(1, k, x/n) &= \sum_{n \leq x^{1/2k}} \mu(n) \Delta(1, k, x/n^{2k}) \ll \\ &\ll \sum_{n \leq x^{1/2k}} \left(\frac{x}{n^{2k}}\right)^{\theta_k} \log^{\theta'_k} x \ll x^{\theta_k} \log^{\theta'_k} x \left(1 + x^{1/2k - \theta_k}\right) \ll x^{\max(\theta_k, 1/2k)} \log^{\theta'_k} x. \end{aligned}$$

So

$$(11) \quad \sum_{n \leq x} \tau^{(r)}(n) = \frac{\zeta(r+1)}{\zeta(2r+2)}x + \frac{\zeta(\frac{1}{r+1})}{\zeta(2)}x^{\frac{1}{r+1}} + O\left(x^{\max(\theta_{r+1}, 1/(2r+2))} \log^{\theta'_{r+1}} x\right).$$

■

For the case $r = 0$ see (3) above.

Lemma 5. *Let $r > 0$, $x^\varepsilon \leq y \leq x^{1/2r}$. Then under RH we have*

$$(12) \quad \mathcal{E}_r(x) = \sum_{n \leq y} \mu(n) \Delta(1, r, x/n^{2r}) + O(x^{1/2+\varepsilon} y^{1/2-r} + x^\varepsilon).$$

Proof. We follow the approach of Montgomery and Vaughan (see [12] or [1]).

First of all consider

$$g_y(s) = \frac{1}{\zeta(s)} - \sum_{d \leq y} \frac{\mu(d)}{d^s}.$$

Then for $\sigma > 1$

$$g_y(s) = \sum_{d > y} \frac{\mu(d)}{d^s}.$$

Assuming RH we have by [15, Th. 14.25]

$$\sum_{d \leq y} \frac{\mu(d)}{d^s} = \zeta^{-1}(s) + O(y^{1/2-\sigma+\varepsilon}(|t|^\varepsilon + 1)) \quad \text{for } \sigma > 1/2 + \varepsilon,$$

so

$$(13) \quad g_y(s) \ll y^{1/2-\sigma+\varepsilon}(|t|^\varepsilon + 1) \quad \text{for } \sigma > 1/2 + \varepsilon.$$

Now let us split $\sum_{n \leq x} \tau^{(r-1)}(n)$ into two parts:

$$\sum_{n \leq x} \tau^{(r-1)}(n) = \sum_{d^{2r} \leq x} \mu(d) T(1, r; x/d^{2r}) = S_1 + S_2,$$

where

$$S_1 := \sum_{d \leq y} \mu(d) T(1, r; x/d^{2r}) = \zeta(r) x \sum_{d \leq y} \frac{\mu(d)}{d^{2r}} + \zeta(1/r) x^{1/r} \sum_{d \leq y} \frac{\mu(d)}{d^2} + \sum_{d \leq y} \mu(d) \Delta(1, r; x/d^{2r})$$

and S_2 is the rest of $\sum_{n \leq x} \tau^{(r-1)}(n)$. We note that under RH by taking into account $y \leq x^{1/2r}$ we have

$$x^{1/r} \sum_{d > y} \frac{\mu(d)}{d^2} \ll x^{1/r} y^{-3/2+\varepsilon} \ll x^{1/2} y^{1/2-r+\varepsilon}$$

and so

$$x^{1/r} \sum_{d \leq y} \frac{\mu(d)}{d^2} = \frac{x^{1/r}}{\zeta(2)} + O(x^{1/2} y^{1/2-r+\varepsilon}).$$

Next, let

$$h_y(s) := \zeta(s) \zeta(rs) g_y(2rs) x^s s^{-1}.$$

Then by Perron formula with $c = 1 + \varepsilon$, $T = x^2$ one can estimate

$$S_2 = \frac{1}{2\pi i} \int_{1+\varepsilon-ix^2}^{1+\varepsilon+ix^2} h_y(s) ds + O(x^\varepsilon).$$

By moving line of integration to $[1/2 + \varepsilon - ix^2, 1/2 + \varepsilon + ix^2]$ we obtain

$$S_2 = \operatorname{res}_{s=1} h(s) + O(I_1 + I_2 + I_3),$$

where

$$I_1 = \int_{1+\varepsilon-ix^2}^{1/2+\varepsilon-ix^2} h(s) ds, \quad I_2 = \int_{1/2+\varepsilon-ix^2}^{1/2+\varepsilon+ix^2} h(s) ds, \quad I_3 = \int_{1/2+\varepsilon-ix^2}^{1+\varepsilon-ix^2} h(s) ds.$$

Due to (13) and estimates of ζ under RH we have

$$\begin{aligned} g_y(2rs) &\ll y^{1/2-r}(|t|^\varepsilon + 1) && \text{for } \sigma > 1/2 + \varepsilon, \\ h(s) &\ll y^{1/2-r}(|t|^\varepsilon + 1) x^s s^{-1} && \text{for } \sigma > 1/2 + \varepsilon, \end{aligned}$$

and

$$\begin{aligned} I_{1,3} &\ll y^{1/2-r+\varepsilon} \max_{\sigma \in [1/2+\varepsilon, 1+\varepsilon]} x^{\sigma-2} \ll y^{1/2-r+\varepsilon}, \\ I_2 &\ll y^{1/2-r+\varepsilon} \int_1^{x^2} x^{1/2} t^{-1} dt \ll y^{1/2-r+\varepsilon} x^{1/2+\varepsilon}. \end{aligned}$$

Identity

$$\operatorname{res}_{s=1} h(s) = \zeta(r) x \sum_{d > y} \frac{\mu(d)}{d^{2r}}$$

completes the proof. ■

Theorem 2. *If Δ is estimated as in (8) and $\theta_r < 1/2r$ then under RH*

$$\mathcal{E}_r(x) = O(x^\alpha), \quad \alpha = \frac{1 - \theta_r}{2r + 1 - 4r\theta_r}.$$

Proof. Let us start with (12):

$$\begin{aligned} \mathcal{E}_r(x) &= \sum_{n \leq y} \mu(n) \Delta(1, r, x/n^{2r}) + O(x^{1/2+\varepsilon} y^{1/2-r} + x^\varepsilon) \ll \sum_{n \leq y} \left(\frac{x}{n^{2r}} \right)^{\theta_r + \varepsilon} + \\ &\quad + O(x^{1/2+\varepsilon} y^{1/2-r} + x^\varepsilon) \ll x^\varepsilon \left(x^{\theta_r} (1 + y^{1-2r\theta_r}) + x^{1/2} y^{1/2-r} + 1 \right). \end{aligned}$$

If $\theta_r < 1/2r$ then

$$\mathcal{E}_r(x) \ll x^\varepsilon \left(x^{\theta_r} y^{1-2r\theta_r} + x^{1/2} y^{1/2-r} \right).$$

Choice $y = x^\beta$, where

$$\beta = \frac{1 - 2\theta_r}{2r + 1 - 4r\theta_r},$$

accomplishes the proof. ■

For the values of θ_b from Table 1 we have

$$\max(\theta_r, 1/2r) = \begin{cases} 1/2r, & r \leq 2, \\ \theta_r, & r > 2. \end{cases}$$

So currently the only non-trivial case of the previous theorem is an estimation for $\tau^{(1)} \equiv \tau^{(e)s}$. We get under assumption of RH that

$$\sum_{n \leq x} \tau^{(1)}(n) = \frac{\zeta(2)}{\zeta(4)} x + \frac{\zeta(1/2)}{\zeta(2)} x^{1/2} + O(x^{\alpha+\varepsilon}),$$

where

$$\alpha = \frac{1 - \theta_2}{5 - 8\theta_2} = \frac{3728}{15469} \approx 0.241 < 1/4.$$

Theorem 3.

$$(14) \quad \mathcal{E}_r(x) = \Omega \left(x^{1/(2r+2)} \right).$$

Proof. Equation (14) is implied by the substitution $m_1 = 1$, $m_2 = r$, $n_1 = 2r$ into Lemma 3. The choice of parameters plainly follows from (9). We obtain

$$\alpha = \frac{1}{2r + 2},$$

which is an exponent in the required Ω -term. ■

5. ASYMPTOTIC PROPERTIES OF $\sum \sigma^{(r)}$

Lemma 6. Let $G_r(s)$ be Dirichlet series for $\sigma^{(r)}$:

$$G_r(s) := \sum_{n=1}^{\infty} \frac{\sigma^{(r)}(n)}{n^s}.$$

Then

$$(15) \quad G_r(s) = \frac{\zeta(s-1)\zeta((r+1)s-r)}{\zeta((r+2)s-r-1)} H_r(s), \quad \sigma > 2,$$

where Dirichlet series $H_r(s)$ converges absolutely for $\sigma > (2r+2)/(2r+3)$.

Proof. Consider Bell series for $\sigma^{(r)}$:

$$\begin{aligned} \sigma_p^{(r)}(x) &:= \sum_{k=0}^{\infty} \sigma^{(r)}(p^k) x^k = \sum_{k=0}^r p^k x^k + \sum_{k>r} (p^r + p^k) x^k = \sum_{k=0}^{\infty} p^k x^k + \sum_{k>r} p^r x^k = \\ &= \frac{1}{1 - px} + \frac{p^r x^{r+1}}{1 - x}. \end{aligned}$$

Then

$$(1 - px)\sigma_p^{(r)}(x) = 1 + \frac{p^r x^{r+1}(1 - px)}{1 - x} = 1 + \sum_{k=0}^{\infty} (p^r x^{r+1+k} - p^{r+1} x^{r+2+k})$$

and

$$\frac{(1 - px)(1 - p^r x^{r+1})}{1 - p^{r+1} x^{r+2}} \sigma_p^{(e)s}(x) = 1 + \frac{p^r x^{r+2}(1 - px)(1 - p^r x^r)}{(1 - x)(1 - p^{r+1} x^{r+2})} := h_p(x).$$

For $\sigma > 1$ we have

$$h_p(p^{-s}) \ll p^{-2}.$$

For $1 \geq \sigma \geq (2r + 2)/(2r + 3) + \varepsilon$ we have

$$h_p(p^{-s}) \ll p^{2r+1-(2r+3)s} \ll p^{-1-\varepsilon}.$$

Now (15) follows from the representation of G_r in the form of infinite product by p :

$$G_r(s) = \prod_p \sigma_p^{(r)}(p^{-s}).$$

■

Following theorem generalizes (4).

Theorem 4.

$$\sum_{n \leq x} \sigma^{(r)}(n) = Dx^2 + O(x \log^{5/3} x), \quad D = \frac{\zeta(r+2)H_r(2)}{2\zeta(r+3)}.$$

Proof. For a fixed r let $z(n)$ be the coefficient at n^{-s} of the Dirichlet series

$$\frac{\zeta(s-1)\zeta((r+1)s-r)}{\zeta((r+2)s-r-1)}$$

and let $h(n)$ be the coefficient of the Dirichlet series $H_r(s)$. It follows from (15) that $\sigma^{(r)} = z \star h$. One can verify that

$$z(n) = \sum_{ab^{r+1}c^{r+2}=n} ab^r c^{r+1} \mu(c).$$

Taking into account Lemma 2 with $(\alpha, \beta) = (r+1, r)$ we obtain

$$\begin{aligned} \sum_{n \leq x} z(n) &= \sum_{c \leq x^{1/(r+2)}} c^{r+1} \mu(c) \left(\frac{\zeta(r+2)}{2} \frac{x^2}{c^{2r+4}} + O(x c^{-r-2} \log^{2/3} x) \right) = \\ &= \frac{\zeta(r+2)}{2\zeta(r+3)} x^2 + O(x \log^{5/3} x) \end{aligned}$$

because

$$\sum_{c \leq x^{1/(r+2)}} \frac{\mu(c)}{c^{r+3}} = \frac{1}{\zeta(r+3)} - \sum_{c > x^{1/(r+2)}} \frac{\mu(c)}{c^{r+3}} = \frac{1}{\zeta(r+3)} + O(x^{-1})$$

and

$$\sum_{c \leq x^{1/(r+2)}} \frac{\mu(c)}{c} \ll \sum_{c \leq x} \frac{1}{c} \ll \log x.$$

Now

$$\begin{aligned} \sum_{n \leq x} \sigma^{(r)}(n) &= \sum_{n \leq x} h(n) \left(\frac{\zeta(r+2)}{2\zeta(r+3)} \frac{x^2}{n^2} + O\left(\frac{x}{n} \log^{5/3} x\right) \right) = \\ &= \frac{\zeta(r+2)}{2\zeta(r+3)} x^2 \sum_{n \leq x} \frac{h(n)}{n^2} + O\left(x \log^{5/3} x \sum_{n \leq x} \frac{h(n)}{n}\right). \end{aligned}$$

But $H_r(s)$ converges absolutely at $\sigma \geq (2r+2)/(2r+3) + \varepsilon$, so

$$\sum_{n \leq x} \frac{h(n)}{n} \ll O(1)$$

and

$$\sum_{n \leq x} \frac{h(n)}{n^2} = H_r(2) - \sum_{n > x} \frac{h(n)}{n^2} = H_r(2) + O(x^{-(2r+4)/(2r+3)+\varepsilon}).$$

■

Theorem 5. For a fixed $r > 0$

$$\sum_{n \leq x} \sigma^{(r)}(n) = Dx^2 + \Omega_{\pm}(x \log x).$$

Proof. The proof almost replicates the proof of [13, Th. 3] with following changes (in notations of [13]):

$$\begin{aligned} \kappa(n) &:= \frac{\sigma^{(r)}(n)}{n}, \\ \sum_{n=1}^{\infty} \frac{\kappa(n)}{n^s} &= \frac{\zeta(s)\zeta((r+1)s+1)}{\zeta((r+2)s+1)} H_r(s+1), \\ v &:= \mu \star \kappa, \\ \sum_{n=1}^{\infty} \frac{v(n)}{n^s} &= \frac{\zeta((r+1)s+1)}{\zeta((r+2)s+1)} H_r(s+1), \\ v(p^a) &= \frac{\sigma^{(r)}(p^a)}{p^a} - \frac{\sigma^{(r)}(p^{a-1})}{p^{a-1}} = \begin{cases} 0, & a \leq r+1, \\ 1/p, & a = r+1, \\ p^{r-a} - p^{r-a+1}, & a > r+1. \end{cases} \end{aligned}$$

We take $m := \log^{1/(4r+4)} x$ and

$$A := \prod_{p \leq m} p^{r+1} \sim e^{(r+1)m} \sim \exp(\log^{1/4} x),$$

then

$$G = \sum_{k \leq u(x)} \frac{v(k)}{k} \gcd(A, k) = \sum_{n^{r+1} | A} v(n^{r+1}) \sum_{k \leq u(x)/n^{r+1}}^* \frac{v(n^{r+1}k)}{v(n^{r+1})k}.$$

Here \sum_k^* means summation over k such that for every $p \mid k$ we have $p \mid n$ or $p \nmid A$. Taking into account $v(p^{r+1}) = 1/p$ we get

$$\begin{aligned} G &= \sum_{n^{r+1} | A} v(n^{r+1}) \sum_{k \geq 1}^* \frac{v(n^{r+1}k)}{v(n^{r+1})k} + o(1) = \\ &= \sum_{n^{r+1} | A} v(n^{r+1}) \prod_{p|n} \left(1 + \sum_{\nu \geq r+2} \frac{v(p^\nu)}{p^{\nu-r-2}} \right) \prod_{p > m} \left(1 + \sum_{\nu \geq r+1} \frac{v(p^\nu)}{p^\nu} \right) + o(1). \end{aligned}$$

Since $|v(p^\nu)| \leq 1/p$ we obtain

$$\sum_{\nu \geq r+1} \frac{v(p^\nu)}{p^\nu} \ll p^{-r-2}.$$

Since $v(n^{r+1}) = 1/n$ for $n^{r+1} \mid A$ and $\log m \asymp \log x$ we have

$$\begin{aligned} G &= (1 + o(1)) \sum_{n^{r+1} \mid A} \frac{1}{n} \prod_{p \mid n} \left(1 + \sum_{\nu \geq r+2} \frac{v(p^\nu)}{p^{\nu-r-2}} \right) = \\ &= (1 + o(1)) \prod_{p \leq m} \left(1 + \frac{1}{p} + \sum_{\nu \geq r+2} \frac{v(p^\nu)}{p^{\nu-r-1}} \right). \end{aligned}$$

But $v(p^\nu) \geq 1/2p^{\nu-r-1}$ for $a \geq r+2$. So

$$\sum_{\nu \geq r+2} \frac{v(p^\nu)}{p^{\nu-r-1}} \geq \sum_{\nu \geq r+2} \frac{1}{2(p^{\nu-r-1})^2} \geq \frac{1}{2p^2}.$$

Hence

$$G \geq (1 + o(1)) \prod_{p \leq m} (1 + p^{-1} + p^{-2}/2) \gg \prod_{p \leq m} (1 + p^{-1}) \gg \log m \gg \log x.$$

■

6. SOME REMARKS

The estimate (5) implies that $\tau^{(r)}(n)/n \rightarrow 0$ as $n \rightarrow \infty$. Thus it is natural to ask what is the maximum value of this ratio.

Lemma 7. *For $n \geq 1$ we have*

$$\tau^{(r)}(n) \leq n,$$

where the equality has place only if $n = 1$ or if $n = 2$ and $r = 0$.

Proof. Recalling the definition (1) we obtain that the least value of a for which $\tau^{(r)}(p^a)$ is different from 1 is $a = r + 1$. So

$$\tau^{(r)}(n) = 2^{\#\{p^{r+1} \mid n\}} \leq 2^{(\log_2 n)/(r+1)} = n^{1/(r+1)}$$

and the statement of the lemma easily follows. ■

One can see that (7) implies

$$\frac{\sigma^{(r)}(n)}{n} \rightarrow +\infty, \quad n \rightarrow \infty.$$

Theorem 6. *Consider the distribution function*

$$S_N(q, r; \lambda) := \frac{1}{N} \#\{n \leq N \mid \sigma^{(r)}(n^q) \leq \lambda n^q\}, \quad q, r \in \mathbb{N}.$$

Then $S_N(q, r; \lambda)$ weakly converges to a function $S(q, r; \lambda)$ which is continuous if and only if $q > r$.

Proof. Let us fix arbitrary q and r and let

$$f(n) := \ln \frac{\sigma^{(r)}(n^q)}{n^q},$$

here f is an additive function. It is enough to prove that

$$F_N(\lambda) := \frac{1}{N} \#\{n \leq N \mid f(n) \leq \lambda\}$$

converges weakly to some $F(\lambda)$ as $N \rightarrow \infty$ and F is continuous if and only if $q > r$.

By definition (2)

$$\sigma^{(r)}(p^q) = \begin{cases} p^q, & r \geq q, \\ p^q + p^r, & r < q. \end{cases}$$

So

$$f(p) = \begin{cases} 0, & r \geq q, \\ \ln(1 + p^{r-q}) \ll p^{r-q}, & r < q, \end{cases}$$

and $f(p) = |f(p)| \leq 1$. Also

$$\sum_p \frac{f(p)}{p} \ll \begin{cases} 0, & r \geq q, \\ \sum_p p^{r-q-1} \ll \sum_p p^{-2}, & r < q, \end{cases} < +\infty.$$

and the same is valid for $\sum_p f^2(p)/p$. Now by Erdős–Wintner theorem [2, Th. i] we get that $F_N(\lambda)$ converges weakly to $F(\lambda)$ as $N \rightarrow \infty$. Taking into account

$$\sum_{f(p) \neq 0} \frac{1}{p} \ll \begin{cases} 0, & r \geq q, \\ \infty, & r < q, \end{cases}$$

we see that due to [2, Th. v] the distribution F is continuous if and only if $r < q$. ■

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